Perturbation Methods in Atmospheric Flight Mechanics

PETER H. ZIPFEL*
Air Force Armament Laboratory, Eglin Air Force Base, Fla.

The classical scalar perturbations of flight mechanics are compared with total vector perturbations. A component perturbation method is introduced which is the generalization of the classical technique, and is valid for all coordinate systems. This method, together with a new tensor formalism, is used to derive the general perturbation equations of atmospheric flight mechanics. These equations hold for any unsteady flight regime and are expressed in a tensor form, invariant under time-dependent coordinate transformations. The perturbation equations of all flight vehicles such as aircraft, missiles, shells, and Magnus rotors, are universally represented by these techniques.

Introduction

THE mathematical modeling techniques of flight mechanics can be grouped into computer simulations and analytical formulations. To evaluate a particular problem by computer the equations of motion are solved in their nonlinear form with only the aerodynamic force functions usually expanded in terms of some small state variables. Despite the use of modern high-speed computers, the analytical techniques still retain their importance to aid the engineer in his design task, providing stability criteria and performance estimates, thus reducing the number of necessary computer runs.

All analytical techniques have in common the requirement for simplified equations of motion suitable for either some stability analysis or closed form solutions. The simplifications, in general, are accomplished with perturbation techniques. The classical small perturbation method was developed to solve specific problems in atmospheric flight mechanics, solely employing scalar perturbations and referring them, for each type of flight vehicle, to a different coordinate system. Also, because simplifications are made before deriving the perturbation equations, the reference flight is limited to steady flight regimes.

The objective of this report is to introduce the general perturbation equations of atmospheric flight mechanics that are valid for all types of flight vehicles and also for unsteady flight regimes. To keep the derivation simple, the flight vehicles are assumed to be rigid bodies.

Tensor Concepts and Notations

To derive the perturbation equations of atmospheric flight dynamics in their most general form, a tensor formalism is required for expressing the dynamic equations as invariants under time-dependent coordinate systems, i.e., the equations hold in every right-handed orthogonal cartesian coordinate system related by time-dependent coordinate transformations. Tensor analysis usually considers space-dependent coordinate transformations only. In flight mechanics the evolution of the vehicle position in time and the motions of the associated frames are of central interest. Time-dependent coordinate transformations therefore dominate the mathematical scene, and it is most desirable to have a time-invariant tensor formulation available.

Such a formalism was developed by Zipfel,[1] and the principal results are summarized below, without proofs, as required for the derivation of the perturbation equations.

Physical objects are modeled by *points* and *frames*. A frame is a generalization of the reference frame and is any set of objects whose mutual distances are time-invariant. It is denoted by a capital letter. A point belonging to this frame is denoted by the same capital letter. *Coordinate systems* are said to be associated with a frame if the components of points, belonging to the frame, are time-invariant in these coordinate systems. A *tensor* is the abstract entity of ordered components that satisfy the transformation law: for a first-order tensor (vector) $[x]$

$$[x]^B = [T]^{BA}[x]^A \tag{1}$$

and for a second-order tensor (tensor) $[X]$

$$[X]^B = [T]^{BA}[X]^A[\tilde{T}]^{BA} \tag{2}$$

The brackets indicate matrix notation. A lower case letter designates a 3×1 matrix and a capital letter a 3×3 matrix. The upper case superscript, e.g., $]^A$, designates the coordinate system associated with frame A. Thus, $[x]^A$ represents the components of a vector in the $]^A$ coordinate system. The abstract entities of a vector and a tensor are written without a superscript, $[x]$ and $[X]$, respectively. The 3×3 coordinate transformation matrix $[T]^{BA}$ denotes the transformation of the $]^B$ coordinate system relative to the $]^A$ coordinate system. The tilde designates the transpose of the matrix. All transformation matrices are orthogonal with their determinants equal to $+1$. Their elements are, in general, functions of time.

Capital subscripts denote points; capital superscripts signify frames. Two subscripts or two superscripts are read from left to right, inserting the word "relative to." For instance, the position of point B relative to point A is designated by the *position vector* $[x_{BA}]$; the orientation of frame B relative to frame A is given by the orthonormal *rotation tensor* $[R^{BA}]$; and the angular velocity of frame B relative to frame A is denoted by $[\omega^{BA}]$ with its skewsymmetric form $[\Omega^{BA}]$.

To formulate the dynamic equations in an invariant form, a new concept, the *rotational time derivative*, was introduced which is invariant under time-dependent coordinate transformations. Note that the ordinary time derivative is only invariant under time-independent coordinate transformations. Symbolically, the rotational time derivative of an arbitrary vector $[x]$ relative to a frame, e.g., A, is written as $[\mathscr{D}^A x]$. For an arbitrary coordinate system $]^B$ its component form is defined as

$$[\mathscr{D}^A x]^B = (d/dt)[x]^B + [T]^{BA}\{(d/dt)[\tilde{T}]^{BA}\}[x]^B \tag{3}$$

Because

$$[\Omega^{BA}]^B = [T]^{BA}(d/dt)[\tilde{T}]^{BA} \tag{4}$$

Equation (3) reduces to the well-known Euler form

$$[\mathscr{D}^A x]^B = (d/dt)[x]^B + [\Omega^{BA}]^B[x]^B \tag{5}$$

A fundamental law, the *generalized Euler theorem*, governs

Presented at the AIAA 2nd Atmospheric Flight Conference, Palo Alto, Calif., September 11–13, 1972; submitted October 2, 1972; revision received May 17, 1973.

Index categories: LV/M Dynamics and Control; LV/M Aerodynamics; Aircraft Aerodynamics (Including Component Aerodynamics).

* Chief, Technology Section. Member AIAA.

the body kinematics. It relates the rotational time derivative of an arbitrary vector $[x]$ relative to two arbitrary frames A and B

$$[\mathscr{D}^A x] = [\mathscr{D}^B x] + [\Omega^{BA}][x] \tag{6}$$

Again, applying Eq. (3) to the first term on the right-hand side, Eq. (6) reduces to the Euler form for the $]^B$ coordinate system, as given by Eq. (5). Thus, behind the classical Euler form of conventional body kinematics lies the rotational time derivative and the generalized Euler theorem of the invariant tensor formalism.

In the remainder of this section are results summarized that follow either from Eq. (3) or (6). The *chain rule*, applied to the product of a tensor $[X]$ and a vector $[x]$, yields for an arbitrary reference frame A

$$[\mathscr{D}^A([X][x])] = [\mathscr{D}^A X][x] + [X][\mathscr{D}^A x] \tag{7}$$

With the aid of the rotational time derivative and the rotation tensor, the angular velocity tensor of frame B relative to frame A can be expressed by

$$[\Omega^{BA}] = [\mathscr{D}^A R^{BA}][\widetilde{R^{BA}}] \tag{8}$$

Some of the useful properties are

$$[\Omega^{CA}] = [\Omega^{CB}] + [\Omega^{BA}] \tag{9}$$

and

$$[\Omega^{BA}] = -[\Omega^{AB}] \tag{10}$$

Finally, the relationship between the rotation tensor and the associated coordinate transformation is established by

$$[R^{BA}]^A = [R^{BA}]^B = [\tilde{T}]^{BA} \tag{11}$$

Perturbation Techniques

The classical perturbation technique of flight dynamics, as for instance outlined by Etkin,[2] proceeds as follows. First, an axis system is defined in relationship to physical quantities, such as the principal body axes or the relative wind velocity. The components of the state variables parallel to these axes are then identified. A particular steady flight regime is selected with certain values for the components, e.g., x_{r1}, x_{r2}, x_{r3}, and a perturbed flight with x_{p1}, x_{p2}, x_{p3}. The differences $\Delta x_i = x_{pi} - x_{ri}$ are the perturbation variables. Because the perturbations are generated by a scalar subtraction of the respective components this technique is also called the scalar perturbation method (see Hopkin[3]).

The disadvantage of the classical perturbation technique lies in the fact that all the formulations are tied to one particular coordinate system. A change to other coordinate systems is very difficult to accomplish.

In theoretical work, vectors are preferred over components and perturbations are defined as the vectorial differences between the reference and perturbed vectors. No allusion is made to a particular coordinate system. Because this technique considers the total state variable rather than its components it is called the *total perturbation method.* For $[x_p]$ and $[x_r]$, the state vectors during the perturbed and reference flight, respectively, the total perturbation is defined as

$$[\delta x] = [x_p] - [x_r] \tag{12}$$

The total perturbations have the advantage over the scalar perturbations in that they hold for any coordinate system. However, in applications, numerical calculations require that vectors are expressed by their components, referred to a particular coordinate system. For instance, vehicle moment-of-inertia is given in body axes, vehicle acceleration and angular velocity are measured by accelerometers and rate gyroscopes, mounted parallel to the body axes; wind-tunnel measurements are recorded in component form, and the whole framework of aerodynamics is based on force and moment components rather than on total values.

To express the total perturbations in components, a transformation matrix must be introduced. In the notation of the previous section, the components of the $[\delta x]$ perturbation, relative to a coordinate system associated with a frame D, become

$$[\delta x]^{Dp} = [x_p]^{Dp} - [T]^{DpDr}[x_r]^{Dr} \tag{13}$$

The subscripts p and r indicate perturbed and reference flights, respectively. $[x_p]^{Dp}$ and $[x_r]^{Dr}$ are the components as they could be measured during the perturbed and the reference flight, and $[T]^{DpDr}$ is the coordinate transformation matrix of the coordinate system associated with the perturbed frame Dp relative to the coordinate system associated with the reference frame Dr.

Every numerical evaluation of equations based on the total perturbation method includes the transformation matrix $[T]^{DpDr}$. Consequently, the transformation angles and their trigonometric functions enter the calculations, and increase considerably the complexity of the equations.

It would be desirable to have a perturbation technique that combines the general invariancy of the total perturbation method for theoretical investigations with the simple component presentation of the scalar perturbation method for practical calculations. Such a technique is obtained if the rotation tensor $[R^{DpDr}]$ of the Dp frame relative to the Dr frame is introduced

$$[\varepsilon x] = [x_p] - [R^{DpDr}][x_r] \tag{14}$$

Equation (14) means that the $[\varepsilon x]$ perturbation is obtained first by rotating the reference state vector through $[R^{DpDr}]$ and then subtracting it from the perturbed state vector. It certainly is expressed in an invariant form. To show that it can be reduced to a simple form for practical calculation, Eq. (14) is written in the $]^{Dp}$ coordinate system

$$\begin{aligned}[\varepsilon x]^{Dp} &= [x_p]^{Dp} - [R^{DpDr}]^{Dp}[x_r]^{Dp} \\ & [x_p]^{Dp} - [R^{DpDr}]^{Dp}[T]^{DpDr}[x_r]^{Dr} \\ &= [x_p]^{Dp} - [x_r]^{Dr}\end{aligned} \tag{15}$$

where the last equality follows from Eq. (11). The transformation matrix of Eq. (13) is absent. Because this technique emphasizes the component form of a vector, it is referred to as the *component perturbation method* or alternately as the ε-perturbations.

In working with the component perturbation method, the choice of the $[R^{DpDr}]$ tensor, and thus the selection of the frame D, is most important. As a general guideline, D is chosen so that the ε-perturbations remain small throughout the flight. Particularly, in atmospheric flight mechanics the selection of D is determined by the requirement of representing the aerodynamic forces as a function of small perturbations. A Taylor series expansion then is possible, and the difficult task of expressing the aerodynamic forces in simple analytical form can be achieved. The designation *"dynamic frame"* will be used for D, because the dynamic equations of flight mechanics are solved in a coordinate system associated with D.

The dynamic frame of airplane dynamics is either the body frame B or the stability frame S. For the definitions of the frames used in this paper, see Etkin.[2] In both cases, for small disturbances, the rotation tensors are close to the unit tensor, expressing the fact that the frame Dp has been rotated by small angles from Dr. For instance, in the simple case of small perturbations about a horizontal steady reference flight, Dr is an inertial frame itself and Dp deviates by the small angles ψ, θ, ϕ from Dr, with α_r included if D is the stability frame. As will be outlined in more detail in the last section, the motions of the dynamic frame relative to the wind frame are directly related to the state variables used in the aerodynamic force expansion.

In missile dynamics, the situation is similar except that the aeroballistic frame replaces the stability frame as dynamic frame. However, for a spinning missile, the body frame cannot serve as a dynamic frame because the perturbations of the aerodynamic roll angle can be large. To keep the perturbations small between the wind and dynamic frame, the nonrolling body frame is chosen as dynamic frame. The motions between the body frame and the dynamic frame thus are not explicitly included in the aerodynamic expansion, but rather the expansion derivatives depend on them implicitly. For simplification, $[R^{DpDr}]$ will be abbreviated by $[R]$ unless ambiguities arise.

The major reason for employing perturbation techniques is the possibility of expanding the aerodynamic force functions in terms of small state variables about the reference flight. Suppose $[f([x])]$ is the force function vector with $[x]$ representing, for simplicity sake, one state vector. The force function during the perturbed flight, $[f([x_p])]$, is expressed, in view of Eq. (14) by

$$[f([x_p])] = [f([\varepsilon x]+[R][x_r])] \tag{16}$$

Expanding about the reference flight, i.e., for $[\varepsilon x] = [0]$, yields

$$[f([x_p])] = [f([R][x_r])]+[\partial f/\partial x][\varepsilon x]+ [\widetilde{\varepsilon x}][\partial^2 f/\partial x^2][\varepsilon x]+\cdots \tag{17}$$

The principle of material indifference states (see Noll)[4] that the physical process of generating aerodynamic forces is independent of spacial attitude. In other words, if $[x_r]$ is rotated through $[R]$ the process or functional dependence is still the same. The only difference being that the force has also been rotated through $[R]$, i.e.,

$$[R][f([x_r])] = [f([R][x_r])] \tag{18}$$

Therefore Eq. (17) becomes

$$[f([x_p])] = [R][f([x_r])]+[\partial f/\partial x][\varepsilon x]+\cdots \tag{19}$$

and $[f]$ is thus also obtained in the form of an ε-perturbation

$$[f_p] = [R][f_r]+[\varepsilon f] \tag{20}$$

This form will be used in the last section for the aerodynamic force expansion.

The component perturbation method satisfies both requirements of invariancy for theoretical investigations and simple component form for practical calculations. It is a generalization of the classical scalar perturbation method and is particularly suited for writing flight mechanical problems in a form invariant under time-dependent coordinate transformations.

Linear and Angular Momentum Equations

The component perturbation method will be used to formulate the general perturbation equations of atmospheric flight mechanics. In this section, the perturbed linear and angular momentum equations will be derived with a detailed account of the aerodynamic force expansion to be followed in the next section.

The linear momentum of the body B with mass m relative to an inertial frame I is defined by

$$[p^{BI}] = m[v_B{}^I] \tag{21}$$

Where $[v_B{}^I]$ is the linear velocity of the center-of-mass B relative to frame I. The angular momentum of body B relative to I and referred to the center-of-mass B is defined by the moment-of-inertia tensor $[I_B{}^B]$ of frame B referred to point B and the angular velocity vector $[\omega^{BI}]$ as follows:

$$[l_B{}^{BI}] = [I_B{}^B][\omega^{BI}] \tag{22}$$

With Eq. (14), the following ε-perturbations are generated

$$[\varepsilon v_B{}^I] = [v_{Bp}{}^I]-[R][v_{Br}{}^I] \tag{23}$$

$$[\varepsilon\omega^{BI}] = [\omega^{BpI}]-[R][\omega^{BrI}] \tag{24}$$

Generalizing these equations for tensors gives

$$[\varepsilon I_B{}^B] = [I_B{}^{Bp}]-[R][I_B{}^{Br}][\tilde{R}] \tag{25}$$

The equations of motion of atmospheric flight mechanics assume the invariant structure

$$[\mathscr{D}^I p^{BI}] = [f_a]+[f_t]+[f_g] \tag{26}$$

$$[\mathscr{D}^I l_B{}^{BI}] = [m_a]+[m_t] \tag{27}$$

Where $[f]$ represents the forces and $[m]$ the moments relative to the center-of-mass B, and the subscripts a, t, and g refer to aerodynamics, thrust, and gravity forces, respectively. Both Eqs. (26) and (27) are valid for the reference and perturbed flights. To derive the linear momentum perturbation equation, let Eq. (26) describe the perturbed flight

$$[\mathscr{D}^I p^{BpI}] = [f_{ap}]+[f_{tp}]+[f_{gp}] \tag{28}$$

and introduce the ε-perturbations for each term

$$[\mathscr{D}^I \varepsilon p^{BI}]+[\mathscr{D}^I([R][p^{BrI}])] = [\varepsilon f_a]+[R][f_{ar}]+[\varepsilon f_t]+ [R][f_{tr}]+[\varepsilon f_g]+[R][f_{gr}] \tag{29}$$

The second term on the left-hand side is modified first by applying the generalized Euler theorem, Eq. (6), the chain rule, Eq. (7), and Eq. (8).

Introducing the skew-symmetric form of the angular velocity perturbations,

$$[\varepsilon\Omega^{DI}] = [\Omega^{DpI}]-[R][\Omega^{DrI}][\tilde{R}] \tag{30}$$

Equation (29) becomes

$$[\mathscr{D}^I \varepsilon p^{BI}]+[\varepsilon\Omega^{DI}][R][p^{BrI}]+ \underline{[R][\mathscr{D}^I p^{BrI}] = [R][f_{ar}]+[R][f_{tr}]+ [R][f_{gr}]}+[\varepsilon f_a]+[\varepsilon f_t]+[\varepsilon f_g] \tag{31}$$

The underlined terms are Eq. (26) for the reference flight, rotated by $[R]$. They are satisfied identically. The gravitational term can be rewritten using the fact that $[f_{gp}] = [f_{gr}]$

$$[\varepsilon f_g] = ([E]-[R])[f_{gr}] \tag{32}$$

Where $[E]$ is the unit tensor.

The perturbation equation of the angular momentum is derived in the same way. Both equations are summarized as follows:

$$[\mathscr{D}^I \varepsilon p^{BI}]+[\varepsilon\Omega^{DI}][R][p^{BrI}] = [\varepsilon f_a]+[\varepsilon f_t]+([E]-[R])[f_{gr}] \tag{33}$$

$$[\mathscr{D}^I \varepsilon l_B{}^{BI}]+[\varepsilon\Omega^{DI}][R][l_B{}^{BrI}] = [\varepsilon m_a]+[\varepsilon m_t] \tag{34}$$

These are *the general perturbation equations of atmospheric flight mechanics.* No small perturbation assumptions have been made as yet. They are expressed in an invariant form, i.e., they hold for all coordinate systems related by time-dependent coordinate transformations. Two types of variables appear: first, $[p^{BrI}]$, $[l_B{}^{BrI}]$, the linear and angular momenta of the reference flight, known as a function of time; second, the component perturbations, marked by a preceding ε. Latter represent the unknowns. The aerodynamic forces and moments will be the subject of the following section. Evaluating the perturbational thrust and gravity forces is straightforward and will not be discussed.

The first terms on the left-hand side of Eqs. (33) and (34) are the time-rate-of-change of linear and angular momenta, respectively; whereas the second terms account for unsteady reference flights. To gain insight into the structure of the perturbation equations two special cases will be discussed.

Case I considers the perturbation equations of aircraft and missiles in steady reference flight. Steady means that the reference flight is unaccelerated and nonrotating. The body frame B is chosen as dynamic frame D. During the reference flight the body frame Br is an inertial frame I. With these substitutions and the fact that $[p^{BrBr}] = [0]$, $[l^{BrBr}] = [0]$, the left-hand sides of the perturbation equations become simply: $[\mathscr{D}^{Br}\varepsilon p^{BBr}]$ and $[\mathscr{D}^{Br}\varepsilon l^{BBr}]$, where

$$[\varepsilon p^{BBr}] = [p^{BpBr}] = m[v_{Bp}{}^{Br}] \tag{35}$$

$$[\varepsilon l_B{}^{BBr}] = [l_B{}^{BpBr}] = [I_{Bp}{}^{Bp}][\omega^{BpBr}] \tag{36}$$

For practical calculations, it is most convenient to express the perturbation equations in a coordinate system associated with the perturbed dynamic frame which is, in this case, the perturbed body frame Bp. The six differential equations thus obtained are the classical perturbation equations of atmospheric flight mechanics. With the assumption that the perturbations and angles are small, the equations reduce to a set of linear differential equations with constant coefficients.

Case II considers the perturbation equations of aircraft and missiles for accelerated but nonrotating reference flight. As in Case I, the body frame B is chosen as dynamic frame. With $[\omega^{BrI}] = [0]$, because of a nonrotating reference flight, Eqs. (33) and (34) simplify to

$$[\mathscr{D}^I \varepsilon p^{BI}]+[\varepsilon\Omega^{BI}][R^{BpBr}][p^{BrI}] = \text{RHS of Eq. (33)} \tag{37}$$

$$[\mathscr{D}^I \varepsilon l_B{}^{BBr}] = \text{RHS of Eq. (34)} \tag{38}$$

Expressed in the $]^{Bp}$ coordinate system Eqs. (37) and (38) become

$$(d/dt)[\varepsilon p^{BI}]^{Bp}+[\Omega^{BpBr}]^{Bp}[\varepsilon p^{BI}]^{Bp}+[\Omega^{BpBr}]^{Bp}[p^{BrI}]^{Br} = [\varepsilon f_a]^{Bp}+[\varepsilon f_t]^{Bp}+([T]^{BpBr}-[E])[f_{gr}]^{Br} \quad (39)$$

$$(d/dt)[\varepsilon l_B^{BBr}]^{Bp}+[\Omega^{BpBr}]^{Bp}[\varepsilon l_B^{BBr}]^{Bp} = [\varepsilon m_a]^{Bp}+[\varepsilon m_t]^{Bp} \quad (40)$$

The last term on the left-hand side of Eq. (39) represents the contribution of the accelerated reference flight. $[p^{BrI}]^{Br}$ is known as a function of time. If small perturbation assumptions are made, the equations can be reduced to a set of linear differential equations with time-dependent coefficients.

Aerodynamic Forces

The most difficult problem in atmospheric flight mechanics is the mathematical modeling of the aerodynamic forces in a form that can be analyzed and evaluated quantitatively. Because the functional form is not known, except for very simple cases, the aerodynamic force functions are expanded in a Taylor series in terms of the state variables relative to a reference flight. Even for digital computer simulations, restrictions for storage and computer time require that the number of independent variables in the aerodynamic tables is kept to a minimum. The dependence on the other variables is then expressed analytically by a Taylor series expansion.

For analytical analysis, the expansion is carried out for all state variables that contribute significantly to the change of the aerodynamic forces. There are two requirements that must be met. First, the partial derivatives of the expansion must be continuous, a condition that is usually satisfied; and second, the expansion variables must be small. In generating the aerodynamic forces, three frames are involved: the atmosphere-fixed frame A, the body frame B, and the wind frame W. If the air is in uniform rectilinear motion relative to an inertial frame, A itself is an inertial frame. The wind frame W has the center-of-mass as one of its points. For detailed definitions, see Etkin.[2] Usually it is postulated that the aerodynamic forces depend on external shape and size, represented by l, atmospheric conditions: density ρ and pressure p; the speed of the body $\|v_B{}^A\|$; the orientation of the body with respect to the wind frame, represented by $[R^{WB}]$ and its derivative $[\mathscr{D}^B R^{WB}]$; the angular velocity of the body relative to air $[\omega^{BA}]$; and finally, the control surface deflections, $[\eta]$. In summary, the functional form is

$$[f_a] = [f_1(l, p, \rho, \|v_B{}^A\|, [R^{WB}], [\omega^{WB}], [\omega^{BA}], [\eta])] \quad (41)$$

Where $[\mathscr{D}^B R^{WB}]$ was substituted by $[\omega^{WB}]$ in view of Eq. (8)

$$[\mathscr{D}^B R^{WB}] = [\Omega^{WB}][R^{WB}] \quad (42)$$

The same functional relationship holds for the aerodynamic moment. The expansion of the aerodynamic force can be carried out in the form of Eq. (41). This is called the *force expansion*. For a discussion of the force expansion and the alternate technique, the coefficient expansion, see Hopkin.[3]

To expand Eq. (41), variables must be identified that remain small throughout the perturbed flight. If the body frame does not yield these variables, a dynamic frame is introduced, like, for instance, in the case of a spinning missile where the nonspinning body frame is used as a dynamic frame. To include all these possibilities, the independent variables in Eq. (41) are separated into two groups

$$[f_a] = [f_2(l, p, \rho, [R^{BD}], [\omega^{BD}]; \|v_B{}^A\|, [R^{WD}], [\omega^{WD}], [\omega^{DA}], [\eta])] \quad (43)$$

The first group, separated by a semicolon, relates the interactions between the body and dynamic frames. Its variables are implicitly included in the expansion. In the case of a spinning missile, e.g., the aerodynamic roll angle and its time derivative belong to this group. For many applications, the method of averaging by Bogoliubov and Mitropolsky[5] can be employed for eliminating this type of dependence.

The second group includes the variables, whose ε-perturbations are to be expanded, for instance

$$[\varepsilon\omega^{WD}] = [\omega^{WpDp}]-[R][\omega^{WrDr}] \quad (44)$$

$$[\varepsilon\eta] = [\eta_p]-[R][\eta_r] \quad (45)$$

It is also more convenient to replace $\|v_B{}^A\|$, $[R^{WD}]$ by $[u^D]$, $[v^D]$, $[w^D]$, the projections of the velocity vector on the dynamic frame axes times the respective unit vector. A sample ε-perturbation is given here

$$[\varepsilon u^D] = [u_p{}^D]-[R][u_r{}^D] \quad (46)$$

To emphasize the procedure, and for simplicity, the expansion of Eq. (43) is carried out in the simpler form

$$[f] = [f([y];\ [x])] \quad (47)$$

Where $[y]$ represents the implicit variables and $[x]$ is the expansion variable. Equation (14) gives the ε-perturbation, and from Eq. (19) the required expansion is obtained

$$[f_p] = [R][f_r([y_p], [x_r])]+[(\partial f/\partial x)([y_p], [R][x_r])][\varepsilon x]+\cdots \quad (48)$$

The force perturbations are therefore

$$[\varepsilon f] = [(\partial f/\partial x)([y_p], [R][x_r])][\varepsilon x]+\cdots \quad (49)$$

and with the variables of Eq. (43)

$$[\varepsilon f_a] = [(\partial f/\partial u^D)\{l, p_p, \rho_p, [R^{BpDp}], [\omega^{BpDp}]; [R][u_r{}^B], \ldots, [R][\omega^{WrDr}], [R][\omega^{DrA}], [R][\eta_r]\}][\varepsilon u^D]+\cdots \quad (50)$$
$$[(\partial f/\partial\omega^{WD})\{\text{same}\}][\varepsilon\omega^{WD}]+[(\partial f/\partial\omega^{DA})\{\text{same}\}][\varepsilon\omega^{DA}]+ [(\partial f/\partial\eta)\{\text{same}\}][\varepsilon\eta]+\text{higher order terms}$$

The same form is obtained for the aerodynamic moment expansion $[\varepsilon m_a]$. If the dynamic frame is selected to be the body frame, as in the case of an aircraft, the implicitly included variables reduce to the parameters of the reference flight.

Up to this point, the derivation was conducted in general tensor notation, i.e., Eq. (50) holds for any coordinate systems related by time-dependent coordinate transformations. For calculations, the coordinate system associated with the perturbed dynamic frame is used to express the aerodynamic force perturbations. The case with frame D = frame B is given as example:

$$[\varepsilon f_a]^{Bp} = [(\partial f/\partial u^B)\{l, p_p\rho_p, [u_r{}^B]^{Br}, \ldots, [\omega^{WrBr}]^{Br}, [\omega^{BrA}]^{Br}, [\eta_r]^{Br}\}]^{Bp}[\varepsilon u^B]^{Bp}+\cdots+[(\partial f/\partial\omega^{WB})\{\text{same}\}]^{Bp}[\varepsilon\omega^{WB}]^{Bp}+ [(\partial f/\partial\omega^{BA})\{\text{same}\}]^{Bp}[\varepsilon\omega^{BA}]^{Bp}+[(\partial f/\partial\eta)\{\text{same}\}]^{Bp}\times [\varepsilon\eta]^{Bp}+\cdots \quad (51)$$

If this equation is normalized, and if p_p and ρ_p can be approximated by p_r and ρ_r, the first three implicit variables are replaced by the Mach and Reynolds numbers of the reference flight. Also $[u_r{}^B]^{Br}$, $[v_r{}^B]^{Br}$, and $[w_r{}^B]^{Br}$ are the velocity components of the reference flight, commonly denoted by u_r, v_r, and w_r; $[\omega^{BrA}]^{Br}$ represents the reference rotations p_r, q_r, and r_r; and $[\omega^{WrBr}]^{Br}$ contains the time-rate-of-change of the aerodynamic angles α_r, β_r. For the simple case of the unaccelerated and nonrotating reference flight, the partial derivatives become constants.

Equation (50) and a similar equation for the aerodynamic moment perturbation, substituted into the linear and angular moment Eqs. (33) and (34), constitute the general perturbation equations of atmospheric flight mechanics.

Conclusions

The theory of perturbations, as presented in this report, is a new approach to the difficult problem of mathematically modeling the dynamics of atmospheric flight vehicles. It unifies the methods employed for different types of vehicles and provides greater insight into the physical aspects of their dynamics. Because the equations are invariant under time-dependent coordinate transformations, they constitute the most general formulation of the perturbation equations of atmospheric flight mechanics. Furthermore, deriving the equations in the compact tensor form saves time and avoids errors. Only the final result need be written in component form for numerical calculations. The new techniques developed in this paper have already been successfully applied to the flight dynamics of Magnus rotors (see Zipfel[1]). These provided the mathematical framework necessary to solve the unsteady lateral stability problem of Magnus rotors in accelerated flight that cannot be solved by the classical perturbation technique. It is hoped that this paper will stimulate attempts to attack as yet unsolved problems and contribute to a better understanding of atmospheric flight mechanics.

References

[1] Zipfel, P. H., "On Flight Dynamics of Magnus Rotors," TR 117 (AD 716 345), Nov. 1970, Dept. of the Army, Fort Detrick, Md.

[2] Etkin, B., *Dynamics of Atmospheric Flight*, Wiley, New York, 1972.

[3] Hopkin, H. R., "A Scheme of Notation and Nomenclature for Aircraft Dynamics and Associated Aerodynamics," TR 66200, June 1966, Royal Aircraft Establishment, England.

[4] Noll, W., "On the Continuity of the Solid and Fluid States," *Journal of Rational Mechanical Analysis*, Vol. 4, No. 1, 1955, p. 19.

[5] Bogoliubov, N. N. and Mitropolsky, Y. A., *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 2nd ed., Gordon and Breach Science Publisher, New York, 1961.

SEPTEMBER 1973 AIAA JOURNAL VOL. 11, NO. 9

Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis

PETER W. LIKINS*
University of California, Los Angeles, Los Angeles, Calif.

AND

FRANK J. BARBERA† AND VICTOR BADDELEY‡
North American Rockwell Corporation, Downey, Calif.

The problem of modal analysis of an elastic appendage on a rotating base is examined with the following objectives: a) to establish the relative advantages of various mathematical models of elastic structures, including the elastic continuum model, the distributed-mass finite-element model, and the concentrated mass model; and b) to extract general inferences concerning the magnitude and character of the influence of spin on the natural frequencies and mode shapes of rotating structures. In realization of the first objective, it is concluded that except for a small class of very special cases the elastic continuum model is barren of useful results, while for constant nominal spin rate the distributed-mass finite-element model is quire generally tractable, since in the latter case the governing equations are always linear, constant-coefficient, ordinary differential equations. Although with both of these alternatives the details of the formulation generally obscure the essence of the problem and permit very little engineering insight to be gained without extensive computation, this difficulty is not encountered when dealing with simple concentrated mass models, which permit determination of the general inferences sought in objective b above.

Introduction

THE literature on flexible spacecraft dynamics is proliferating at a rate which reflects the serious concern of the aerospace community for this problem. Many investigators in this field now employ some system of hybrid coordinates for dynamic analysis, using a combination of discrete coordinates (for the translations and rotations of rigid bodies or reference frames) and distributed or modal coordinates (for the deformations of elastic bodies). Although various idealizations have been adopted for mathematical models of deformable vehicles or vehicle appendages, including elastic continua,[1-7] distributed-mass finite-element systems,[8] and elastically interconnected nodal body systems,[9-15] in every case when modal coordinates are employed some rationale for the selection and truncation of these coordinates must be established.

The purpose of this paper is to address the problems of mathematical modeling and modal coordinate selection for an elastic appendage attached to a rigid base which is constrained to rotate with a constant angular speed Ω about a body-axis fixed in inertial space. As shown in many of the references (e.g., Ref. 8), the modal coordinates appropriate for fully constrained base rotation are often also appropriate for a hybrid coordinate representation of deformations of an elastic appendage attached to a rigid base which freely maintains the nominal constant angular velocity when the appendage deformation remains at its constant, steady-state value, but which differs slightly from the nominal constant angular velocity due to appendage deformational perturbations.

Although the restriction to a constant nominal base motion is formally necessary for the development of a rational policy of coordinate selection, it may be expected that experienced engineers will find the results of this special case applicable informally to a wider range of engineering problems than we indicate here.

Modal analysis of an idealized vibrating elastic structure on a rotating base requires the derivation of the linearized equations of small oscillatory deviation of the mathematical model from

Received October 30, 1972; revision received April 23, 1973. Research supported by NASA Marshall Space Flight Center under Contract NAS 8-26214.

Index category: Structural Dynamic Analysis.

*Professor, Mechanics and Structures Department. Associate Fellow AIAA.

† Member of the Technical Staff; formerly UCLA graduate student. Member AIAA.

‡ Member of the Technical Staff.